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Original Article

# Some results associated with the max–min and min–max compositions of bifuzzy matrices

**E.G. Emam***, **M.A. Fndh**

*Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt*

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**Abstract** In this paper, we define some kinds of bifuzzy matrices, the max–min ($\circ$) and the min–max ($*$) compositions of bifuzzy matrices are defined. Also, we get several important results by these compositions. However, we construct an idempotent bifuzzy matrix from any given one through the min–max composition.

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## 1. Introduction

The concept of bifuzzy sets (or intuitionistic fuzzy sets) was introduced by Atanassov [1] as a generalization of fuzzy subsets. Later on, much fundamental works have done with this concept by Atanassov [2,3] and others [4–7]. A bifuzzy relation is a pair of fuzzy relations, namely, a membership and a non-membership function, which represent positive and negative aspects of the given information. This is why the concept of bifuzzy relations is a generalization of the idea of fuzzy relations. The name “bifuzzy relations” is used for objects introduced by Atanassov and originally called intuitionistic fuzzy relations (see [1,2]). Bifuzzy relations are also called by some authors “bipolar fuzzy relations” (see [6]). Since the concept of bifuzzy relations is an extension for the concept of ordinary fuzzy relations, the concept of bifuzzy matrices (which represent finite bifuzzy relations) is also an extension for the concept of ordinary fuzzy matrices.

In this paper, we study and prove some properties of bifuzzy matrices throughout some compositions of these matrices. However, we concentrate our attention for the two compositions $\circ$ (*max–min*) and its dual composition $*$ (*min–max*). We use the definitions of some kinds of bifuzzy matrices such as nearly constant, symmetric, nearly irreflexive and others to prove some results. One of these results enables us to construct an idempotent bifuzzy matrix from any bifuzzy matrix and this is the main result in our work. We also state the relationship between the two compositions $\circ$ and $*$ of bifuzzy matrices.The motivation for this paper is to study some kinds of finite bifuzzy relations

* Corresponding author. Tel.: +201002854283.
E-mail addresses: eg_emom@yahoo.com (E.G. Emam), m.fndh@yahoo.com (M.A. Fndh).
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throughout bifuzzy matrices by using the two compositions $\circ$ and $*$.

## 2. Preliminaries and definitions

In system models which based on fuzzy sets, one often uses fuzzy matrices (matrices with elements having values anywhere in the closed interval [0, 1]) to define finite fuzzy relations.

When the related universes $X$ and $Y$ of a fuzzy relation $R$ are finite such that $|X| = m, |Y| = n$, a fuzzy matrix $R = [r_{ij}]_{m\times n}$ whose generic term $r_{ij} = \mu_R(x_i, y_j)$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$ where the function $\mu_R : X \times Y \to [0, 1]$ is called the membership function and $r_{ij}$ is the grade of membership of the element $(x_i, y_j)$ in $R$.

**Definition 2.1** [8,9]**.** Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times l}$ be two fuzzy matrices. Then the *max–min* composition ($\circ$) of $A$ and $B$ is denoted by $A \circ B$ and is defined as

$$A \circ B = [t_{ij}]_{m\times l} = \bigvee_{k=1}^{n} (a_{ik} \wedge b_{kj}).$$

The *min–max* composition ($*$) of $A$ and $B$ is denoted by $A^*B$ and is defined as

$$A * B = [s_{ij}]_{m\times l} = \bigwedge_{k=1}^{n} (a_{ik} \vee b_{kj}),$$

where $\vee$, $\wedge$ are the maximum and minimum operations respectively.

**Definition 2.2** (bifuzzy matrix [6,10,11])**.** Let $A' = [a'_{ij}]_{m\times n}, A'' = [a''_{ij}]_{m\times n}$ be two fuzzy matrices such that $a'_{ij} + a''_{ij} \le 1$ for every $i \le m, j \le n$. The pair $(A', A'')$ is called a bifuzzy matrix and we may write $A = [a_{ij} = \langle a'_{ij}, a''_{ij}\rangle]_{m\times n}$. The numbers $a'_{ij}$ and $a''_{ij}$ denote the degree of membership and the degree of non-membership of the $ij^{\text{th}}$ element in $A$ respectively. Thus the bifuzzy matrix $A$ takes its elements from the set $F = \{< a', a'' >: a', a'' \in [0, 1], a' + a'' \le 1\}$

For each bifuzzy matrix $A$ of kind $m \times n$, there is a fuzzy matrix $\pi_A$ associated with $A$ such that $\pi_{ij} = 1 - a'_{ij} - a''_{ij}$ for every $i \le m, j \le n$. The number $\pi_{ij}$ is called the degree of indeterminacy of the $ij^{\text{th}}$ element in $A$ or called the degree of hesitancy of $ij^{\text{th}}$ element in $A$. It is obvious that $0 \le \pi_{ij} \le 1$ for every $i \le m, j \le n$. Especially, if $\pi_{ij} = 0$ for all $i \le m, j \le n$, then the bifuzzy matrix $A$ is reduced to the ordinary fuzzy matrix. Thus fuzzy matrices are special cases from bifuzzy matrices.

Now, we define some operations on the set $F$ defined above.
For $a =< a', a'' >, b =< b', b'' >\in F$, we define:
$a \wedge b =< min(a', b'), max(a'', b'') >$,
$a \vee b =< max(a', b'), min(a'', b'') >$,
$a^c =< a'', a' >$ and $a \le b$ if and only if $a' \le b', a'' \ge b''$,

$$a \ominus b = \begin{cases} < 0, a'' > & \text{if } \ a' \le b', a'' < b'', \\ < 0, 1 > & \text{if } \ a' \le b', a'' \ge b'', \\ < a', a'' > & \text{if } a' > b'. \end{cases}$$

We may write $\mathbf{0}$ instead of $< 0, 1 >$ and $\mathbf{1}$ instead of $<1, 0>$.
For the bifuzzy matrices $A = [a_{ij} = \langle a'_{ij}, a''_{ij}\rangle]_{n\times n}$, $B = [b_{ij} = \langle b'_{ij}, b''_{ij}\rangle]_{n\times n}$ and $C = [c_{ij} = \langle c'_{ij}, c''_{ij}\rangle]_{n\times m}$, let us define the following matrix operations [8–11].

$A \wedge B = [a_{ij} \wedge b_{ij}]$,
$A \vee B = [a_{ij} \vee b_{ij}]$,
$A \ominus B = [a_{ij} \ominus b_{ij}]$,

$$A * C = \left[\left\langle \bigwedge_{k=1}^{n} (a'_{ik} \vee c'_{kj}), \bigvee_{k=1}^{n} (a''_{ik} \wedge c''_{kj})\right\rangle\right],$$

$$A \circ C = \left[\left\langle \bigvee_{k=1}^{n} (a'_{ik} \wedge c'_{kj}), \bigwedge_{k=1}^{n} (a''_{ik} \vee c''_{kj})\right\rangle\right].$$

For simplictly write $AC$ instead of $A\circ C$. However,
$A^k = A^{k-1}A$, where
$A^k = \left[a^{(k)}_{ij} = \langle a'^{(k)}_{ij}, a''^{(k)}_{ij}\rangle\right] = A^{k-1}A$ and

$$I_n = A^0 = \begin{cases} \mathbf{1} & \text{if } i = j, \\ \mathbf{0} & \text{if } i \ne j. \end{cases},$$

$A^t = [a_{ji}]$ (the transpose of $A$),
$A^c = [a_{ji} = \langle a''_{ij}, a'_{ij}\rangle]$ (the complement of $A$),
$A \le B$ if and only if $a_{ij} \le b_{ij}$ for every $i, j. \le n$.

## 3. Theoretical results of the paper

**Definition 3.1** (reflexive, irreflexive bifuzzy matrix [6,8,9,11])**.** An $n \times n$ bifuzzy matrix $A = [a_{ij}]$ is called reflexive (irreflexive) if and only if $a_{ii} = \mathbf{1}$ ($a_{ii} = \mathbf{0}$). It is also called weakly reflexive (nearly irreflexive) if and only if $a_{ii} \ge a_{ij}$ ($a_{ii} \le a_{ij}$) for every $i, j \le n$.

**Lemma 3.2.** *Let $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$ be two nearly irreflexive bifuzzy matrices. Then $A * B \le A \vee B$.*

**Proof.** Let $R = A * B$ and $T = A \vee B$. Then $r_{ij} = \left\langle \bigwedge_{k=1}^{n} (a'_{ik} \vee b'_{kj}), \bigvee_{k=1}^{n} (a''_{ik} \wedge b''_{kj})\right\rangle$ and $t_{ij} =< a'_{ij} \vee b'_{ij}, a''_{ij} \wedge b''_{ij} >$ . Now,

$$r'_{ij} = \bigwedge_{k=1}^{n} (a'_{ik} \vee b'_{kj}) \le a'_{ii} \vee b'_{ij} \le a'_{ij} \vee b'_{ij} = t'_{ij} \text{ and}$$

$$r''_{ij} = \bigvee_{k=1}^{n} (a''_{ik} \wedge b''_{kj}) \ge a''_{ii} \wedge b''_{ij} \ge a''_{ij} \wedge b''_{ij} = t''_{ij}.$$ Thus, we have $r_{ij} \le t_{ij}$ and so $A * B \le A \vee B$. □

It is noted that $A \vee B = B$ for $A \le B$.

**Lemma 3.3.** *Let $A$ and $B$ be two nearly irreflexive bifuzzy matrices and $A \le B$. Then $A * B \le B$.*

**Proof.** By Lemma 3.2. □

**Definition 3.4** (symmetric, asymmetric bifuzzy matrix [6,9])**.** An $n \times n$ bifuzzy matrix $A = [a_{ij}]$ is called symmetric if and only if $A = A^t$ and it is also called asymmetric if and only if $a_{ij} \wedge a_{ji} = \mathbf{0}$ for every $i, j \le n$.

**Remark.** It should be noted that any asymmetric bifuzzy matrix is also irreflexive.

**Proposition 3.5.** *Let $A = [a_{ij=}\langle a'_{ij}, a''_{ij}\rangle]_{n\times n}$ be a symmetric and nearly irreflexive bifuzzy matrix. Then we have:*

(1) $A * A \le A$,
(2) *$A * A$ is symmetric and nearly irreflexive,*
(3) *$A^2$ is weakly reflexive.*

**Proof.** (1) By Lemmas 3.2 and 3.3.

(2) Suppose $S = A * A$. It is obvious that $S$ is symmetric and so

$$s'_{ii} = \bigwedge_{k=1}^{n} (a'_{ik} \vee a'_{ki}) = \bigwedge_{k=1}^{n} a'_{ik} \le \bigwedge_{k=1}^{n} (a'_{ik} \vee a'_{kj}) = s'_{ij}$$

and

$$s''_{ii} = \bigvee_{k=1}^{n} (a''_{ik} \wedge a''_{ki}) = \bigvee_{k=1}^{n} a''_{ik.} \ge \bigvee_{k=1}^{n} (a''_{ik} \wedge a''_{kj}) = s''_{ij}.$$

Thus, $s_{ii} \le s_{ij}$ and so that$S$ is nearly irreflexive.
(3) Let $T = A^2$. Then

$t_{ij} = \left\langle \bigvee_{k=1}^{n} (a'_{ik} \wedge a'_{kj}), \bigwedge_{k=1}^{n} (a''_{ik} \vee a''_{kj}) \right\rangle$, i.e.,

$t'_{ij} = \bigvee_{k=1}^{n} (a'_{ik} \wedge a'_{kj}) = a'_{ih} \wedge a'_{hj}$ for some $h \leq n$.

But since $A$ is symmetric, we have

$t'_{ii} = \bigvee_{k=1}^{n} (a'_{ik} \wedge a'_{ki}) = \bigvee_{k=1}^{n} a'_{ik} \geq a'_{ih} \geq a'_{ih} \wedge a'_{hj} = t'_{ij}$.

Also,

$t''_{ij} = \bigwedge_{k=1}^{n} (a''_{ik} \vee a''_{kj}) = a''_{is} \vee a''_{sj}$ for some $s \leq n$

and

$t''_{ii} = \bigwedge_{k=1}^{n} (a''_{ik} \vee a''_{ki}) = \bigwedge_{k=1}^{n} a''_{ik} \leq a''_{is} \leq a''_{is} \vee a''_{sj} = t''_{ij}$. That is $t_{ii} \geq t_{ij}$

and $A^2$ is thus weakly reflexive. □

**Remark.** We notice that the bifuzzy matrix $A * A$ is symmetric and irreflexive when $A$ is also so.

**Proposition 3.6.** *For bifuzzy matrices* $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$, $C = [c_{ij}]_{n\times l}$ *and* $D = [d_{ij}]_{p\times m}$, *we have:*

(1) $(B * C)^t = C^t * B^t$,
(2) *If* $A \leq B$, *then* $D * A \leq D * B$ *and* $A * C \leq B * C$.

**Proof.** (1) Let $S = C^t * B^t$ and $T = B * C$. Then

$s_{ij} = \left\langle \bigwedge_{k=1}^{n} (c'_{ki} \vee b'_{jk}), \bigvee_{k=1}^{n} (c''_{ki} \wedge b''_{jk}) \right\rangle$ and

$t_{ji} = \left\langle \bigwedge_{k=1}^{n} (b'_{jk} \vee c'_{ki}), \bigvee_{k=1}^{n} (b''_{jk} \wedge c''_{ki}) \right\rangle$,

i.e., $S = T^t$.

(2) Let $W = D * A$ and $G = D * B$, i.e.,

$w_{ij} = \left\langle \bigwedge_{k=1}^{m} (d'_{ik} \vee a'_{kj}), \bigvee_{k=1}^{m} (d''_{ik} \wedge a''_{kj}) \right\rangle$

and

$g_{ij} = \left\langle \bigwedge_{k=1}^{m} (d'_{ik} \vee b'_{kj}), \bigvee_{k=1}^{m} (d''_{ik} \wedge b''_{kj}) \right\rangle$.

Since we have that $A \leq B$, we get $a'_{kj} \leq b'_{kj}$ and $a''_{kj} \geq b''_{kj}$ and so $d'_{ik} \vee a'_{kj} \leq d'_{ik} \vee b'_{kj}$ and $d''_{ik} \wedge a''_{kj} \geq d''_{ik} \wedge b''_{kj}$ for every $k \leq m$. Therefore,

$\bigwedge_{k=1}^{m} (d'_{ik} \vee a'_{kj}) \leq \bigwedge_{k=1}^{m} (d'_{ik} \vee b'_{kj})$ and $\bigvee_{k=1}^{m} (d''_{ik} \wedge a''_{kj}) \geq \bigvee_{k=1}^{m} (d''_{ik} \wedge b''_{kj})$,

i.e., $w_{ij} \leq g_{ij}$.

Similarly, one can show that $A * C \leq B * C$. □

**Theorem 3.7.** *For any* $m \times n$ *bifuzzy matrix* $A$, $A * A^t$ *is nearly irreflexive and symmetric.*

**Proof.** Let $R = A * A^t$. That is

$r_{ij} = \left\langle \bigwedge_{k=1}^{n} (a'_{ik} \vee a'_{jk}), \bigvee_{k=1}^{n} (a''_{ik} \wedge a''_{jk}) \right\rangle$, i.e.,

$r'_{ij} = \bigwedge_{k=1}^{n} (a'_{ik} \vee a'_{jk}) = a'_{il} \vee a'_{jl}$ for some $l \leq n$

and

$r''_{ij} = \bigvee_{k=1}^{n} (a''_{ik} \wedge a''_{jk}) = a''_{ig} \wedge a''_{jg}$ for some $g \leq n$.

Now,

$r'_{ii} = \bigwedge_{k=1}^{n} (a'_{ik} \vee a'_{ik}) = \bigwedge_{k=1}^{n} a'_{ik} = a'_{ih}$ and $r''_{ii} = \bigvee_{k=1}^{n} (a''_{ik} \wedge a''_{ik}) = a''_{im}$ for some $h, m \leq n$.

Since $r'_{ii} = a'_{ih} \leq a'_{il} \leq a'_{il} \vee a'_{jl} = r'_{ij}$ and $r''_{ii} = a''_{im} \geq a''_{ig} \geq a''_{ig} \wedge a''_{jg} = r''_{ij}$, we get $r_{ii} \leq r_{ij}$ and $A^* A^t$ is nearly irreflexive. The symmetry of $R$ is obvious. □

**Corollary 3.8.** *For any* $m \times n$ *bifuzzy matrix* $A$, *we have:*

(1) $(A * A^t) * (A * A^t) \leq A * A^t$,
(2) $(A * A^t) * (A * A^t)$ *is symmetric and nearly irreflexive,*
(3) $(A * A^t)^2$ *is weakly reflexive.*

**Proof.** By Proposition 3.5 and Theorem 3.7. □

**Proposition 3.9.** *Let* $A$ *be an* $n \times n$ *asymmetric bifuzzy matrix. Then* $A * A^t = O$ *(the zero matrix)*

**Proof.** Let $T = A * A^t$. Then

$t_{ij} = \left\langle \bigwedge_{k=1}^{n} (a'_{ik} \vee a'_{jk}), \bigvee_{k=1}^{n} (a''_{ik} \wedge a''_{jk}) \right\rangle$

$= \left\langle a'_{ih} \vee a'_{jh}, a''_{is} \wedge a''_{js} \right\rangle$ for some $h, s \leq n$.

Notice that, since $A$ is asymmetric, it is irreflexive and so $t'_{ij} = a'_{ih} \vee a'_{jh} \leq a'_{ij} \vee a'_{jj} = a'_{ij}$ and $t''_{ij} = a''_{is} \wedge a''_{js} \geq a''_{ij} \wedge a''_{jj} = a''_{ij}$. That is $t_{ij} \leq a_{ij}$.

Similarly, we can see that $t_{ij} \leq a_{ji}$ and $t_{ij} \leq a_{ij} \wedge a_{ji} = \mathbf{0}$. Thus, $t_{ij} = \mathbf{0}$ and so $T = O$. □

**Definition 3.10** (nilpotent, transitive, idempotent bifuzzy matrix [8,9,11]). An $n \times n$ bifuzzy matrix $A$ is called nilpotent if and only if $A^n = O$ (the zero matrix), it is also called transitive if and only if $A^2 \leq A$ and it is called idempotent if and only if $A^2 = A$.

**Proposition 3.11** ([11], pp. 224). *If* $A$ *is nilpotent, then* $A^m$ *is irreflexive for every* $m \leq n$.

*The following proposition shows that the nilpotency of a bifuzzy matrix* $A$ *implies the asymmetry of that matrix. However, the converse is not always true.*

**Proposition 3.12.** *Let* $A$ *be an* $n \times n$ *nilpotent bifuzzy matrix. Then* $A$ *is asymmetric.*

**Proof.** Since $A$ is nilpotent, $a_{ij}^{(n)} = \mathbf{0}$.

If $a_{ij} \wedge a_{ji} > \mathbf{0}$, i.e., if $a'_{ij} \wedge a'_{ji} > 0$ and $a''_{ij} \vee a''_{ji} < 1$, then $a'_{ij} > 0$, $a'_{ji} > 0$, $a''_{ij} < 1$ and $a''_{ji} < 1$.

Now, we have two cases for $n$.

**Case 1:** If $n$ is odd, then

$a'^{(n)}_{ij} \geq \underbrace{a'_{ij} \wedge a'_{ji} \wedge a'_{ij} \wedge \ldots \wedge a'_{ij}}_{\text{n - elements}} > 0$ and

$a''^{(n)}_{ij} \leq \underbrace{a''_{ij} \vee a''_{ji} \vee a''_{ij} \vee \ldots \vee a''_{ij}}_{\text{n - elements}} < 1$

which contradicts the nilpotency of $A$.

**Case 2:** If $n$ is even, then by Proposition 3.11 we have

$a'^{(n)}_{ii} \geq \underbrace{a'_{ij} \wedge a'_{ji} \wedge a'_{ij} \wedge \ldots \wedge a'_{ji}}_{\text{n - elements}} > 0$

and

$a''^{(n)}_{ii} \leq \underbrace{a''_{ij} \vee a''_{ji} \vee a''_{ij} \vee \ldots \vee a''_{ji}}_{\text{n - elements}} < 1$

which is also, a contradiction. Thus, $a'_{ij} \wedge a'_{ji} = 0$ and $a''_{ij} \vee a''_{ji} = 1$. That is $a_{ij} \wedge a_{ji} = \mathbf{0}$ and $A$ is then asymmetric. □

**Proposition 3.13** ([11], pp. 222). *If* $A$ *is irreflexive and transitive bifuzzy matrix, then* $A$ *is nilpotent.*

**Proposition 3.14.** *Let* $A$ *and* $B$ *be two transitive bifuzzy matrices, such that* $A \leq B$. *Then* $A \ominus B^t$ *is transitive*

**Proof.** Let $D = A \ominus B^t$ and suppose $d_{ik} \wedge d_{kj} = c > \mathbf{0}$ for some $k \leq n$. That is

$(\langle a'_{ik}, a''_{ik} \rangle \ominus \langle b'_{ki}, b''_{ki} \rangle) \wedge (\langle a'_{kj}, a''_{kj} \rangle \ominus \langle b'_{jk}, b''_{jk} \rangle) = \langle c', c'' \rangle > \langle 0, 1 \rangle$. Thus, $a'_{ik} > b'_{ki}$ and $a'_{kj} > b'_{jk}$. So that $\langle a'_{ik}, a''_{ik} \rangle \wedge \langle a'_{kj}, a''_{kj} \rangle = \langle c', c'' \rangle$, i.e., $a'_{ik} \wedge a'_{kj} = c'$ and $a''_{ik} \vee a''_{kj} = c''$.

Since $A$ is transitive,

$a_{ij} = \langle a'_{ij}, a''_{ij}\rangle \geq \left\langle a'_{ik} \wedge a'_{kj}, a''_{ik} \vee a''_{kj}\right\rangle = \langle c', c''\rangle.$

Now, we show that if $a'_{ij} \leq b'_{ji}$, there are contradictions. (a) If $a'_{ik} = c'$, then $b'_{ki} < c'$ and so $a'_{ki} < c'$ (since we have that $A \leq B$). However, since we have assumed $b'_{ji} \geq a'_{ij} \geq c'$, we get

$b'_{ki} \geq b'_{kj} \wedge b'_{ji} \geq a'_{kj} \wedge b'_{ji} \geq c'$.

Which is a contradiction.

(b) If $a'_{kj} = c'$, then $b'_{jk} < c'$. However, $b'_{jk} \geq b'_{ji} \wedge b'_{ik} \geq b'_{ji} \wedge a'_{ik} \geq c'$.Which is also a contradiction.

Therefore, $a'_{ij} > b'_{ji}$ and so

$$d_{ij} = a_{ij} \ominus b_{ji} = \langle a'_{ij}, a''_{ij}\rangle \ominus \langle b'_{ji}, b''_{ji}\rangle$$
$$= \langle a'_{ij}, a''_{ij}\rangle \geq \left\langle a'_{ik} \wedge a'_{kj}, a''_{ik} \vee a''_{kj}\right\rangle = \langle c', c''\rangle,$$

i.e., $d_{ij} \geq c = d_{ik} \wedge d_{kj}$ and $D$ is thus transitive. This completes the proof. □

**Corollary 3.15.** *Let $A$ and $B$ be two transitive bifuzzy matrices, with $A \leq B$. Then $(A \ominus B^t) * (A \ominus B^t)^t = O$.*

**Proof.** It is easy to see that $A \ominus B^t$ is irreflexive and so by Propositions 3.9, 3.12–3.14, we get the result. □

**Definition 3.16** (constant, nearly constant bifuzzy matrix [8], pp. 84)**.** An $m \times n$ bifuzzy matrix $A = [a_{ij}]$ is called constant if and only if $a_{ij} = a_{kj}$ for every $i, k \in \{1, 2, ..., m\}, j \in \{1, 2, ..., n\}$, $A$ is nearly constant if and only if $a_{ij} = a_{kj}$, where $i \neq j$ for every $k \neq j$.

**Theorem 3.17.** *Let $S$ be an $n \times n$ symmetric and nearly irreflexive bifuzzy matrix. Then the bifuzzy matrix $T = I_n * S$ is idempotent and nearly constant.*

**Proof.** Based on the symmetry of $S$, we can write the elements of the bifuzzy matrix $T$ in terms of the elements of $S$ as follows:

$$t_{ij} = \langle t'_{ij}, t''_{ij}\rangle = \begin{cases} \langle s'_{jj}, s''_{jj}\rangle & \text{if } i \neq j, \\ \left\langle \bigwedge_{i\neq k} s'_{ik}, \bigvee_{i\neq k} s''_{ik}\right\rangle & \text{if } i = j. \end{cases}$$

First, from the definition of $t_{ij}$, we notice that $T$ is nearly constant. Now, we will show that $T$ is idempotent. Any element $t^{(2)}_{ij}$ of $T^2$ is calculated as:

$$t^{(2)}_{ij} = \left\langle t'^{(2)}_{ij}, t''^{(2)}_{ij}\right\rangle = \left\langle \bigvee_{k=1}^{n} (t'_{ik} \wedge t'_{kj}), \bigwedge_{k=1}^{n} (t''_{ik} \vee t''_{kj})\right\rangle$$
$$= \left\langle t'_{ih} \wedge t'_{hj}, t''_{il} \vee t''_{lj}\right\rangle \text{ for some } h, l \leq n.$$

However, we have several cases for the indices $i, j, h$ and $l$ to show that $t^{(2)}_{ij} = t_{ij}$.

**Case 1:** Suppose that $i = j = h = l$. In this case we have

$t'^{(2)}_{ij} = t'_{ih} \wedge t'_{hj} = t'_{ij} \wedge t'_{ij} = t'_{ij}$

and

$t''^{(2)}_{ij} = t''_{il} \vee t''_{lj} = t''_{ij} \vee t''_{ij} = t''_{ij}$.

Thus, $t^{(2)}_{ij} = t_{ij}$.

**Case 2:** Suppose that $i = j = h \neq l$. In this case we have

$t'^{(2)}_{ij} = t'_{ij}$ as in Case 1. Also,

$t''^{(2)}_{ij} = t''_{il} \vee t''_{lj} \leq t''_{ij} \vee t''_{jj} = t''_{ij} \vee t''_{ij} = t''_{ij}$ (since we have $i = j$).

On the other hand, since we have that $S$ is nearly irreflexive,

$$t''_{ij} = \bigvee_{i\neq k} s''_{ik} \leq s''_{ii} \leq s''_{ll} \vee s''_{ii} = s''_{ll} \vee s''_{jj} = t''_{il} \vee t''_{lj} = t''^{(2)}_{ij}.$$

Thus, $t''^{(2)}_{ij} = t''_{ij}$ and so $t^{(2)}_{ij} = t_{ij}$.

**Case 3:** Suppose that $i = j = l \neq h$. In this case we have

$t'^{(2)}_{ij} = t'_{ih} \wedge t'_{hj} \geq t'_{ij} \wedge t'_{jj} \geq t'_{ij} \wedge t'_{ij} = t'_{ij}$ (since $i = j$).

But

$$t'_{ij} = \bigwedge_{i\neq k} s'_{ik} \geq s'_{ii} \geq s'_{hh} \wedge s'_{ii} = s'_{hh} \wedge s'_{jj} = t'_{ih} \wedge t'_{hj} = t'^{(2)}_{ij}.$$

Thus, $t'^{(2)}_{ij} = t'_{ij}$.

Also, as in Case 1, we get $t''^{(2)}_{ij} = t''_{ij}$, hence $t^{(2)}_{ij} = t_{ij}$.

**Case 4:** Suppose that $i = h = l \neq j$. In this case we have

$t'_{ii} \wedge t'_{ij} = t'_{ih} \wedge t'_{hj} \geq t'_{ij} \wedge t'_{jj}$ and so $t'_{ii} \geq t'_{jj}$. But by the definition of $t'_{jj}$, it is clear that $t'_{jj} \geq s'_{jj}$ (since $S$ is nearly irreflexive) so that $t'_{ii} \geq t'_{jj} \geq s'_{jj}$.

Thus, $t'^{(2)}_{ij} = t'_{ih} \wedge t'_{hj} = t'_{ii} \wedge t'_{hj} = t'_{ii} \wedge s'_{jj} = s'_{jj} = t'_{ij}$.

Also, in this case we have $t''_{ii} \vee t''_{ij} = t''_{il} \vee t''_{lj} \leq t''_{ij} \vee t''_{jj}$ and so $t''_{ii} \leq t''_{jj}$.

But $t''_{jj} \leq s''_{jj}$ (since $S$ is nearly irreflexive) and so $t''_{ii} \leq t''_{jj} \leq s''_{jj}$.

Thus, $t''^{(2)}_{ij} = t''_{il} \vee t''_{lj} = t''_{ii} \vee t''_{lj} = t''_{ii} \vee s''_{jj} = s''_{jj} = t''_{ij}$.

Therefore, $t^{(2)}_{ij} = t_{ij}$.

**Case 5:** Suppose that $j = l = h \neq i$. In this case we have

$$t'^{(2)}_{ij} = t'_{ih} \wedge t'_{hj} = t'_{ij} \wedge t'_{jj} = s'_{jj} \wedge (\bigwedge_{j\neq k} s'_{jk}) = s'_{jj} = t'_{ij}$$

and

$$t''^{(2)}_{ij} = t''_{il} \vee t''_{lj} = t''_{ij} \vee t''_{jj} = s''_{jj} \vee (\bigvee_{j\neq k} s''_{jk}) = s''_{jj} = t''_{ij}.$$

**Case 6:** Suppose that $i = j \neq h \neq l$. In this case we have

$t'^{(2)}_{ij} = t'_{ih} \wedge t'_{hj} \geq t'_{ij} \wedge t'_{jj} = t'_{ij} \wedge t'_{ij} = t'_{ij}$ (since $i = j$). On the other hand, since we have $S$ is nearly irreflexive

$$t'_{ij} = \bigwedge_{i\neq k} s'_{ik} \geq s'_{ii} \geq s'_{hh} \wedge s'_{ii} = s'_{hh} \wedge s'_{jj} = t'_{ih} \wedge t'_{hj} = t'^{(2)}_{ij}.$$

Thus, $t'^{(2)}_{ij} = t'_{ij}$.

Also,

$t''^{(2)}_{ij} = t''_{il} \vee t''_{lj} \leq t''_{ij} \vee t''_{jj} = t''_{ij}$

and

$$t''_{ij} = \bigvee_{i\neq k} s''_{ik} \leq s''_{ii} \leq s''_{hh} \vee s''_{ii} = s''_{hh} \vee s''_{jj} = t''_{ih} \vee t''_{hj} = t''^{(2)}_{ij}.$$

Thus, $t''^{(2)}_{ij} = t''_{ij}$ and so $t^{(2)}_{ij} = t_{ij}$.

**Case 7:** Suppose that $i = h \neq j \neq l$. In this case we have

$t'_{ii} \wedge t'_{ij} = t'_{ih} \wedge t'_{hj} \geq t'_{ij} \wedge t'_{jj}$ and so $t'_{ii} \geq t'_{jj} \geq s'_{jj}$. As in Case 4, we get $t'^{(2)}_{ij} = t'_{ij}$.

Also,

$s''_{ll} \vee s''_{jj} = t''_{il} \vee t''_{lj} \leq t''_{ij} \vee t''_{jj} = s''_{jj} \vee (\bigvee_{j\neq k} s''_{jk}) = s''_{jj}$ (since $S$ is nearly irreflexive). So, $s''_{ll} \leq s''_{jj}$. Therefore,

$t''^{(2)}_{ij} = t''_{il} \vee t''_{lj} = s''_{ll} \vee s''_{jj} = s''_{jj} = t''_{ij}$.

Thus, $t^{(2)}_{ij} = t_{ij}$.

**Case 8:** Suppose that $i = l \neq h \neq j$. In this case we have

$$t'^{(2)}_{ij} = t'_{ih} \wedge t'_{hj} \geq t'_{ij} \wedge t'_{jj} = s'_{jj} \wedge (\bigwedge_{j\neq k} s'_{jk}) = s'_{jj} = t'_{ij}.$$

On the other hand we have

$t'_{ij} = s'_{jj} \geq s'_{hh} \wedge s'_{jj} = t'_{ih} \wedge t'_{hj} = t'^{(2)}_{ij}$.

Therefore, $t'^{(2)}_{ij} = t'_{ij}$.

Also, as in Case 4, we get $t''^{(2)}_{ij} = t''_{ij}$. Thus, $t^{(2)}_{ij} = t_{ij}$.

**Case 9:** Suppose that $j = h \neq i \neq l$. In this case, we have

$$t'^{(2)}_{ij} = t'_{ih} \wedge t'_{hj} = t'_{ij} \wedge t'_{jj} = s'_{jj} \wedge (\bigwedge_{j\neq k} s'_{jk}) = s'_{jj} = t'_{ij}$$

and

$t''^{(2)}_{ij} = t''_{ij}$, as in Case 7. Therefore, $t^{(2)}_{ij} = t_{ij}$.

**Case 10:** Suppose that $j = l \neq h \neq i$. In this case, we have

$$t'^{(2)}_{ij} = t'_{ih} \wedge t'_{hj} \geq t'_{ij} \wedge t'_{jj} = s'_{jj} \wedge (\bigwedge_{j\neq k} s'_{jk}) = s'_{jj} = t'_{ij}.$$

On the other hand, $t'_{ij} = s'_{jj} \geq s'_{hh} \wedge s'_{jj} = t'_{ih} \wedge t'_{hj} = t'^{(2)}_{ij}$.

Thus, $t'^{(2)}_{ij} = t'_{ij}$.

Also,

$$t''^{(2)}_{ij} = t''_{il} \vee t''_{lj} = t''_{ij} \vee t''_{jj} = s''_{jj} \vee (\bigvee_{j\neq k} s''_{jk}) = s''_{jj} = t''_{ij}.$$

Therefore, $t^{(2)}_{ij} = t_{ij}$.

**Case 11:** Suppose that $h = l \neq i \neq j$. As in Case 8, $t'^{(2)}_{ij} = t'_{ij}$ and $t''^{(2)}_{ij} = t''_{ij}$ as in Case 7. Therefore, $t^{(2)}_{ij} = t_{ij}$.

**Case 12:** Suppose that $i \neq j \neq h \neq l$. As in Cases 4 and 9, $t^{(2)}_{ij} = t_{ij}$. From the computations of $t^{(2)}_{ij}$, we find that $t^{(2)}_{ij} = t_{ij}$ in all the above cases and so $T$ is idempotent. □

**Corollary 3.18.** *Let $A$ be any $m \times n$ bifuzzy matrix. Then the matrix $I_m * (A * A^t)$ is idempotent and nearly constant.*

**Proof.** By Theorems 3.7 and 3.17. □

**Corollary 3.19.** *Let $A$ be any $m \times n$ bifuzzy matrix. Then the matrix $(A * A^t) * I_m$ is idempotent.*

**Proof.** Notice that $((A * A^t)^t * I_m))^t = I_m * (A * A^t)$. Then by Corollary 3.18, the bifuzzy matrix $((A * A^t)^t * I_m))^t$ is idempotent. So $(A * A^t)^t * I_m$ is idempotent. But $(A * A^t)^t = A * A^t$. Thus, $(A * A^t) * I_m$ is idempotent. □

Theorem 3.17 and its corollaries are useful in studying bifuzzy relations (bifuzzy matrices). However, they enable us to construct an idempotent bifuzzy relation (matrix) from any given bifuzzy relation (matrix).

**Example.** Let

$$A = \begin{bmatrix} \langle 0.5, 0.3\rangle & \langle 0.4, 0.6\rangle & \langle 0.8, 0.2\rangle & \langle 0.7, 0.3\rangle \\ \langle 0.8, 0\rangle & \langle 0.9, 0.1\rangle & \langle 1, 0\rangle & \langle 0.3, 0.6\rangle \\ \langle 0.6, 0.4\rangle & \langle 0.5, 0.5\rangle & \langle 0, 1\rangle & \langle 0.8, 0.1\rangle \\ \langle 0.7, 0.2\rangle & \langle 0.6, 0.3\rangle & \langle 0.9, 0\rangle & \langle 0.5, 0.4\rangle \end{bmatrix}.$$

Then

$$\begin{aligned} S &= A * A^t \\ &= \begin{bmatrix} \langle 0.5, 0.3\rangle & \langle 0.4, 0.6\rangle & \langle 0.8, 0.2\rangle & \langle 0.7, 0.3\rangle \\ \langle 0.8, 0\rangle & \langle 0.9, 0.1\rangle & \langle 1, 0\rangle & \langle 0.3, 0.6\rangle \\ \langle 0.6, 0.4\rangle & \langle 0.5, 0.5\rangle & \langle 0, 1\rangle & \langle 0.8, 0.1\rangle \\ \langle 0.7, 0.2\rangle & \langle 0.6, 0.3\rangle & \langle 0.9, 0\rangle & \langle 0.5, 0.4\rangle \end{bmatrix} \\ &\quad * \begin{bmatrix} \langle 0.5, 0.3\rangle & \langle 0.8, 0\rangle & \langle 0.6, 0.4\rangle & \langle 0.7, 0.2\rangle \\ \langle 0.4, 0.6\rangle & \langle 0.9, 0.1\rangle & \langle 0.5, 0.5\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.8, 0.2\rangle & \langle 1, 0\rangle & \langle 0, 1\rangle & \langle 0.9, 0\rangle \\ \langle 0.7, 0.3\rangle & \langle 0.3, 0.6\rangle & \langle 0.8, 0.1\rangle & \langle 0.5, 0.4\rangle \end{bmatrix} \\ &= \begin{bmatrix} \langle 0.4, 0.6\rangle & \langle 0.7, 0.3\rangle & \langle 0.5, 0.5\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.7, 0.3\rangle & \langle 0.3, 0.6\rangle & \langle 0.8, 0.1\rangle & \langle 0.5, 0.4\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.8, 0.1\rangle & \langle 0, 1\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.6, 0.3\rangle & \langle 0.5, 0.4\rangle & \langle 0.6, 0.3\rangle & \langle 0.5, 0.4\rangle \end{bmatrix}. \end{aligned}$$

It is clear that $S$ is nearly irreflexive and symmetric. Also, let $T = I_4 * S$. That is

$$\begin{aligned} T &= \begin{bmatrix} \langle 1, 0\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 1, 0\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 1, 0\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 1, 0\rangle \end{bmatrix} \\ &\quad * \begin{bmatrix} \langle 0.4, 0.6\rangle & \langle 0.7, 0.3\rangle & \langle 0.5, 0.5\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.7, 0.3\rangle & \langle 0.3, 0.6\rangle & \langle 0.8, 0.1\rangle & \langle 0.5, 0.4\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.8, 0.1\rangle & \langle 0, 1\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.6, 0.3\rangle & \langle 0.5, 0.4\rangle & \langle 0.6, 0.3\rangle & \langle 0.5, 0.4\rangle \end{bmatrix} \\ &= \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.3, 0.6\rangle & \langle 0, 1\rangle & \langle 0.5, 0.4\rangle \\ \langle 0.4, 0.6\rangle & \langle 0.5, 0.4\rangle & \langle 0, 1\rangle & \langle 0.5, 0.4\rangle \\ \langle 0.4, 0.6\rangle & \langle 0.3, 0.6\rangle & \langle 0.5, 0.5\rangle & \langle 0.5, 0.4\rangle \\ \langle 0.4, 0.6\rangle & \langle 0.3, 0.6\rangle & \langle 0, 1\rangle & \langle 0.5, 0.4\rangle \end{bmatrix}. \end{aligned}$$

Then it is obvious that $T$ is nearly constant and one can show that it is also idempotent by calculating $T^2$.

**Lemma 3.20.** *For $a, b \in F$, we have:*
*(1) $(a \vee b)^c = a^c \wedge b^c$,(2) $(a \wedge b)^c = a^c \vee b^c$. The proof is trivial.*

The following proposition shows the relationship between the two composition $*$ and $\circ$ of bifuzzy matrices.

**Proposition 3.21.** *For bifuzzy matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times l}$, we have:*

(1) $(A * B)^c = A^c B^c$,
(2) $A^c * B^c = (AB)^c$.

**Proof.** (1) Let $R = (A * B)^c$ and $D = A^c B^c$. Then

$$\begin{aligned} r_{ij} &= \left\langle \bigwedge_{k=1}^{n} (a'_{ik} \vee b'_{kj}), \bigvee_{k=1}^{n} (a''_{ik} \wedge b''_{kj}) \right\rangle^c \\ &= \left\langle \bigvee_{k=1}^{n} (a''_{ik} \wedge b''_{kj}), \bigwedge_{k=1}^{n} (a'_{ik} \vee b'_{kj}) \right\rangle \end{aligned}$$

and

$$d_{ij} = \left\langle \bigvee_{k=1}^{n} (a''_{ik} \wedge b''_{kj}), \bigwedge_{k=n}^{n} (a'_{ik} \vee b'_{kj}) \right\rangle.$$

Therefore, $R = D$.

(2) Similarly, we can show that $A^c * B^c = (AB)^c$. □

**Corollary 3.22.** *For bifuzzy matrices $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{n\times p}$, $C = [c_{ij}]_{p\times g}$ and $D = [d_{ij}]_{m\times p}$, we have:*

(1) $A * (B * C) = (A * B) * C$,
(2) $(A * B) = D$ *if and only if* $A^c B^c = D^c$ .

From the above corollary, it is seen that the operation $*$ is associative. We will prove that $*$ is distributive over the operations $\vee$ and $\wedge$ in the following proposition.

**Proposition 3.23.** *For any three bifuzzy matrices $A$, $B$ and $C$ of order $m \times n$, $n \times m$ and $n \times m$ respectively, we have:*

(1) $A * (B \vee C) = (A * B) \vee (A * C)$,
(2) $A * (B \wedge C) = (A * B) \wedge (A * C)$.

**Proof.** (1) Let $D = B \vee C$, $R = A * D$, $G = A * B$, $H = A * C$ and $W = G \vee H$. Then

$$d_{ij} = \langle b'_{ij} \vee c'_{ij}, b''_{ij} \wedge c''_{ij} \rangle,$$

$$r_{ij} = \left\langle \bigwedge_{k=1}^{n} (a'_{ik} \vee d'_{kj}), \bigvee_{k=1}^{n} (a''_{ik} \wedge d''_{kj}) \right\rangle$$

$$= \left\langle \bigwedge_{k=1}^{n} \left(a'_{ik} \vee (b'_{kj} \vee c'_{kj})\right), \bigvee_{k=1}^{n} \left(a''_{ik} \wedge (b''_{kj} \wedge c''_{kj})\right) \right\rangle,$$

$$g_{ij} = \left\langle \bigwedge_{k=1}^{n} (a'_{ik} \vee b'_{kj}), \bigvee_{k=1}^{n} (a''_{ik} \wedge b''_{kj}) \right\rangle$$

and

$$h_{ij} = \left\langle \bigwedge_{k=1}^{n} (a'_{ik} \vee c'_{kj}), \bigvee_{k=1}^{n} (a''_{ik} \wedge c''_{kj}) \right\rangle.$$

Thus,

$$\begin{aligned} w_{ij} &= g_{ij} \vee h_{ij} \\ &= \left\langle \left( \bigwedge_{k=1}^{n} (a'_{ik} \vee b'_{kj}) \right) \vee \left( \bigwedge_{k=1}^{n} (a'_{ik} \vee c'_{kj}) \right), \left( \bigvee_{k=1}^{n} (a''_{ik} \wedge b''_{kj}) \right) \right. \\ &\qquad \left. \wedge \left( \bigvee_{k=1}^{n} (a''_{ik} \wedge c''_{kj}) \right) \right\rangle \\ &= \left\langle \bigwedge_{k=1}^{n} \left( (a'_{ik} \vee b'_{kj}) \vee (a'_{ik} \vee c'_{kj}) \right), \right. \\ &\qquad \left. \bigvee_{k=1}^{n} \left( (a''_{ik} \wedge b''_{kj}) \wedge (a''_{ik} \wedge c''_{kj}) \right) \right\rangle \\ &= \left\langle \bigwedge_{k=1}^{n} \left(a'_{ik} \vee (b'_{kj} \vee c'_{kj})\right), \bigvee_{k=1}^{n} \left(a''_{ik} \wedge (b''_{kj} \wedge c''_{kj})\right) \right\rangle. \end{aligned}$$

We conclude that $A * (B \vee C) = (A * B) \vee (A * C)$.

(2) Can be proved by similar manner. □

**Proposition 3.24.** *For bifuzzy matrices $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{n\times p}$ and $C = [c_{ij}]_{n\times p}$, we have:*

(1) $A*(B\ominus C)\geq(A*B)\ominus(A*C)$,
(2) $A(B\ominus C)=AB\ominus AC$.

**Proof.** (1) Let $R=B\ominus C, D=A*R, S=A*B, E=A*C$ and $H=S\ominus E$. Then

$$r_{ij}=\begin{cases}\langle 0,b''_{ij}\rangle & \text{if } b'_{ij}\leq c'_{ij}, b''_{ij}<c''_{ij},\\ \langle 0,1\rangle & \text{if } b'_{ij}\leq c'_{ij}, b''_{ij}\geq c''_{ij},\\ \langle b'_{ij},b''_{ij}\rangle & \text{if } \ b'_{ij}>c'_{ij}.\end{cases},$$

$$s_{ij}=\left\langle\bigwedge_{k=1}^{n}(a'_{ik}\vee b'_{kj}),\ \bigvee_{k=1}^{n}(a''_{ik}\wedge b''_{kj})\right\rangle$$

and

$$e_{ij}=\left\langle\bigwedge_{k=1}^{n}(a'_{ik}\vee c'_{kj}),\ \bigvee_{k=1}^{n}(a''_{ik}\wedge c''_{kj})\right\rangle.$$

Thus,

$$d_{ij}=\begin{cases}\left\langle\bigwedge_{k=1}^{n}a'_{ik},\ \bigvee_{k=1}^{n}(a''_{ik}\wedge b''_{kj})\right\rangle & \text{if } \ b'_{ij}\leq c'_{ij}, b''_{ij}<c''_{ij},\\ \left\langle\bigwedge_{k=1}^{n}a'_{ik},\ \bigvee_{k=1}^{n}a''_{ik}\right\rangle & \text{if } \ b'_{ij}\leq c'_{ij}, b''_{ij}\geq c''_{ij},\\ \left\langle\bigwedge_{k=1}^{n}(a'_{ik}\vee b'_{kj}),\ \bigvee_{k=1}^{n}(a''_{ik}\wedge b''_{kj})\right\rangle & \text{if } \ b'_{ij}>c'_{ij}.\end{cases}$$

and

$$h_{ij}=\begin{cases}\left\langle 0,\ \bigvee_{k=1}^{n}(a''_{ik}\wedge b''_{kj})\right\rangle\\ \quad\text{if } \bigwedge_{k=1}^{n}(a'_{ik}\vee b'_{kj})\leq\bigwedge_{k=1}^{n}(a'_{ik}\vee c'_{kj}),\\ \quad\quad \bigvee_{k=1}^{n}(a''_{ik}\wedge b''_{kj})<\bigvee_{k=1}^{n}(a''_{ik}\wedge c''_{kj}),\\ \langle 0,1\rangle\\ \quad\text{if } \ \bigwedge_{k=1}^{n}(a'_{ik}\vee b'_{kj})\leq\bigwedge_{k=1}^{n}(a'_{ik}\vee c'_{kj}),\\ \quad\quad \bigvee_{k=1}^{n}(a''_{ik}\wedge b''_{kj})\geq\bigvee_{k=1}^{n}(a''_{ik}\wedge c''_{kj}),\\ \left\langle\bigwedge_{k=1}^{n}(a'_{ik}\vee b'_{kj}),\ \bigvee_{k=1}^{n}(a''_{ik}\wedge b''_{kj})\right\rangle\\ \quad\text{if } \ \bigwedge_{k=1}^{n}(a'_{ik}\vee b'_{kj})>\bigwedge_{k=1}^{n}(a'_{ik}\vee c'_{kj}).\end{cases}$$

i.e.,

$$h_{ij}=\begin{cases}\left\langle 0,\ \bigvee_{k=1}^{n}(a''_{ik}\wedge b''_{kj})\right\rangle & \text{if } b'_{kj}\leq c'_{kj}, b''_{kj}<c''_{kj},\\ \langle 0,1\rangle & \text{if } b'_{kj}\leq c'_{kj}, b''_{kj}\geq c''_{kj},\\ \left\langle\bigwedge_{k=1}^{n}(a'_{ik}\vee b'_{kj}),\ \bigvee_{k=1}^{n}(a''_{ik}\wedge b''_{kj})\right\rangle & \text{if } b'_{kj}>c'_{kj}.\end{cases}$$

We note that $D\geq H$. Hence $A*(B\ominus C)\geq(A*B)\ominus(A*C)$.
(2) Similar to (1). □

This proposition shows that the operation $\circ$ is distributive over the operation $\ominus$.

**Proposition 3.25.** *For bifuzzy matrices* $A=[a_{ij}]_{m\times n}$, $B=[b_{ij}]_{n\times l}$, $C=[c_{ij}]_{y\times p}$, $D=[d_{ij}]_{p\times m}$ *and* $E=[e_{ij}]_{l\times g}$, *we have:*

(1) $C(D*A*B)E\leq CD*A*BE$,
(2) $(C*D)A(B*E)\leq C*(DAB)*E$.

**Proof.** (1) Let $Q=D*A*B$, $T=CQ$ and $R=TE$. That is $R$ is the left-hand side of the inequality. Also, let $S=CD$, $H=BE$, $G=S*A$ and $W=G*H$. That is $W$ is the right-hand side of the inequality. Then

$$q_{ij}=\left\langle\bigwedge_{x=1}^{n}\left[\left(\bigwedge_{u=1}^{m}(d'_{iu}\vee a'_{ux})\right)\vee b'_{xj}\right],\ \bigvee_{x=1}^{n}\left[\left(\bigvee_{u=1}^{m}(d''_{iu}\wedge a''_{ux})\right)\wedge b''_{xj}\right]\right\rangle.$$

Thus,

$$\begin{aligned}t'_{ij}&=\bigvee_{k=1}^{p}(c'_{ik}\wedge q'_{kj})\\&=\bigvee_{k=1}^{p}\left(c'_{ik}\wedge\left\{\bigwedge_{x=1}^{n}\left[\left(\bigwedge_{u=1}^{m}(d'_{ku}\vee a'_{ux})\right)\vee b'_{xj}\right]\right\}\right)\end{aligned}$$

and

$$\begin{aligned}t''_{ij}&=\bigwedge_{k=1}^{p}(c''_{ik}\vee q''_{kj})\\&=\bigwedge_{k=1}^{p}\left(c''_{ik}\vee\left\{\bigvee_{x=1}^{n}\left[\left(\bigvee_{u=1}^{m}(d''_{ku}\wedge a''_{ux})\right)\wedge b''_{xj}\right]\right\}\right).\end{aligned}$$

Thus, we have

$$\begin{aligned}r'_{ij}&=\bigvee_{v=1}^{l}(t'_{iv}\wedge e'_{vj})\\&=\bigvee_{v=1}^{l}\left(\left\{\bigvee_{k=1}^{p}\left(c'_{ik}\wedge\left\{\bigwedge_{x=1}^{n}\left[\left(\bigwedge_{u=1}^{m}(d'_{ku}\vee a'_{ux})\right)\vee b'_{xv}\right]\right\}\right)\right\}\wedge e'_{vj}\right)\\&=\bigvee_{v=1}^{l}\bigvee_{k=1}^{p}\bigwedge_{x=1}^{n}\bigwedge_{u=1}^{m}\left(c'_{ik}\wedge(d'_{ku}\vee a'_{ux}\vee b'_{xv})\wedge e'_{vj}\right)\\&=\bigvee_{v=1}^{l}\bigvee_{k=1}^{p}\bigwedge_{x=1}^{n}\bigwedge_{u=1}^{m}((c'_{ik}\wedge d'_{ku}\wedge e'_{vj})\vee(c'_{ik}\wedge a'_{ux}\wedge e'_{vj})\\&\quad\vee(c'_{ik}\wedge b'_{xv}\wedge e'_{vj}))\end{aligned}$$

and

$$\begin{aligned}r''_{ij}&=\bigwedge_{v=1}^{l}(t''_{iv}\vee e''_{vj})\\&=\bigwedge_{v=1}^{l}\left(\left\{\bigwedge_{k=1}^{p}\left(c''_{ik}\vee\left\{\bigvee_{x=1}^{n}\left[\left(\bigvee_{u=1}^{m}(d''_{ku}\wedge a''_{ux})\right)\wedge b''_{xv}\right]\right\}\right)\right\}\vee e''_{vj}\right)\\&=\bigwedge_{v=1}^{l}\bigwedge_{k=1}^{p}\bigvee_{x=1}^{n}\bigvee_{u=1}^{m}\left(c''_{ik}\vee(d''_{ku}\wedge a''_{ux}\wedge b''_{xv})\vee e''_{vj}\right).\end{aligned}$$

Thus,

$$s_{ij}=\left\langle\bigvee_{k=1}^{p}(c'_{ik}\wedge d'_{kj}),\ \bigwedge_{k=1}^{p}(c''_{ik}\vee d''_{xj})\right\rangle,$$

$$\begin{aligned}g_{ij}&=\left\langle\bigwedge_{u=1}^{m}(s'_{iu}\vee a'_{uj}),\ \bigvee_{u=1}^{m}(s''_{iu}\wedge a''_{uj})\right\rangle\\&=\left\langle\bigwedge_{u=1}^{m}\left(\left[\bigvee_{k=1}^{p}(c'_{ik}\wedge d'_{ku})\right]\vee a'_{uj}\right),\ \bigvee_{u=1}^{m}\left(\left[\bigwedge_{k=1}^{p}(c''_{ik}\vee d''_{xu})\right]\wedge a''_{uj}\right)\right\rangle\end{aligned}$$

and

$$h_{ij}=\left\langle\bigvee_{v=1}^{l}(b'_{iv}\wedge e'_{vj}),\ \bigwedge_{v=1}^{l}(b''_{iv}\vee e''_{vj})\right\rangle.$$

Thus,

$$\begin{aligned}w'_{ij}&=\bigwedge_{x=1}^{n}(g'_{ix}\vee h'_{xj})\\&=\bigwedge_{x=1}^{n}\left(\left\{\bigwedge_{u=1}^{m}\left[\left(\bigvee_{k=1}^{p}(c'_{ik}\wedge d'_{ku})\right)\vee a'_{ux}\right]\right\}\vee\left\{\bigvee_{v=1}^{l}(b'_{xv}\wedge e'_{vj})\right\}\right)\\&=\bigwedge_{x=1}^{n}\bigwedge_{u=1}^{m}\bigvee_{k=1}^{p}\bigvee_{v=1}^{l}\left((c'_{ik}\wedge d'_{ku})\vee a'_{ux}\vee(b'_{xv}\wedge e'_{vj})\right)\\&=\bigvee_{v=1}^{l}\bigvee_{k=1}^{p}\bigwedge_{u=1}^{m}\bigwedge_{x=1}^{n}\left((c'_{ik}\wedge d'_{ku})\vee a'_{ux}\vee(b'_{xv}\wedge e'_{vj})\right)\end{aligned}$$

and

$$\begin{aligned}w''_{ij}&=\bigvee_{x=1}^{n}(g''_{ix}\wedge h''_{xj})\\&=\bigvee_{x=1}^{n}\left(\left\{\bigvee_{u=1}^{m}\left[\left(\bigwedge_{k=1}^{p}(c''_{ik}\vee d''_{ku})\right)\wedge a''_{ux}\right]\right\}\wedge\left\{\bigwedge_{v=1}^{l}(b''_{xv}\vee e''_{vj})\right\}\right)\\&=\bigvee_{x=1}^{n}\bigvee_{u=1}^{m}\bigwedge_{k=1}^{p}\bigwedge_{v=1}^{l}\left((c''_{ik}\vee d''_{ku})\wedge a''_{ux}\wedge(b''_{xv}\vee e''_{vj})\right)\\&=\bigwedge_{v=1}^{l}\bigwedge_{k=1}^{p}\bigvee_{u=1}^{m}\bigvee_{x=1}^{n}((c''_{ik}\wedge a''_{ux}\wedge b''_{xv})\vee(c''_{ik}\wedge a''_{ux}\wedge e''_{vj})\\&\quad\vee(d''_{ku}\wedge a''_{ux}\wedge b''_{xv})\vee(d''_{ku}\wedge a''_{ux}\wedge e''_{vj})).\end{aligned}$$

Since $c'_{ik}\wedge d'_{ku}\wedge e'_{vj}\leq c'_{ik}\wedge d'_{ku}$, $c'_{ik}\wedge a'_{ux}\wedge e'_{vj}\leq a'_{ux}$ and $c'_{ik}\wedge b'_{xv}\wedge e'_{vj}\leq b'_{xv}\wedge e'_{vj}$, we get $r'_{ij}\leq w'_{ij}$. Also, since $c''_{ik}\wedge a''_{ux}\wedge b''_{xv}\leq c''_{ik}$, $d''_{ku}\wedge a''_{ux}\wedge e''_{vj}\leq e''_{vj}$ and $c''_{ik}\wedge a''_{ux}\wedge e''_{vj}\leq e''_{vj}$, we get $r''_{ij}\geq w''_{ij}$. Thus, $r_{ij}\leq w_{ij}$ and $R\leq W$.
(2) Similar to (1). □

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